On Certain Projections of C^*-Matrix Algebras

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Abstract

H. Dye defined the projections $P_{i,j}(a)$ of a C^*-matrix algebra by

$$
\begin{aligned}
P_{i,j}(a) &= (1 + aa^*)^{-1} \otimes E_{i,i} + (1 + aa^*)^{-1}a \otimes E_{i,j} \\
&+ a^*(1 + aa^*)^{-1} \otimes E_{j,i} + a^*(1 + aa^*)^{-1}a \otimes E_{j,j},
\end{aligned}
$$

and he used it to show that in the case of factors not of type I_{2n}, the unitary group determines the algebraic type of that factor. We study these projections and we show that in $\mathbb{M}_2(\mathbb{C})$, the set of such projections includes all the projections. For infinite C^*-algebra A, having a system of matrix units, including the Cuntz algebra $\mathcal{O}_n$, we have $A \simeq \mathbb{M}_n(A)$. M. Leen proved that in a simple, purely infinite C^*-algebra A, the $*$-symmetries generate $\mathcal{U}_0(A)$. We revise and modify Leen's proof to show that part of such $*$-isometry factors are of the form $1 - 2P_{i,j}(\omega)$, $\omega \in \mathcal{U}(A)$. In simple, unital purely infinite C^*-algebras having trivial K_1-group, we prove that all $P_{i,j}(\omega)$ have trivial K_0-class. In particular, if $u \in \mathcal{U}(\mathcal{O}_n)$, then u can be factorized as a product of $*$-symmetries, where eight of them are of the form $1 - 2P_{i,j}(\omega)$.

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1 Introduction

Let A be a unital C^*-algebra. The set of projections and the group of unitaries of A are denoted by $\mathcal{P}(A)$ and $\mathcal{U}(A)$, respectively. Recall that the C^*-matrix algebra over A which is denoted by $\mathbb{M}_n(A)$ is the algebra of all $n \times n$ matrices $(a_{i,j})$ over A, with the usual addition, scalar multiplication, and multiplication of matrices and the involution (adjoint) is $(a_{i,j})^* = (a_{j,i}^*)$. As in Dye's viewpoint of $\mathbb{M}_n(A)$, let $S_n(A)$ denote the direct sum of n copies of A, considered as a left A-module. Addition of n-tuples $\bar{x} = (x_1, x_2, \ldots, x_n)$ in $S_n(A)$ is componentwise

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and $a \in A$ acts on $\bar{x}$ by $a(\bar{x}) = (ax_1, ax_2, \ldots, ax_n)$. Then $S_n(A)$ is a Hilbert C^*-algebra module, with the inner product defined by

$$< \bar{x}, \bar{y} > = \sum_{i=1}^{n} x_i y_i^*.$$

By an A-endomorphism T of $S_n(A)$, we mean an additive mapping on $S_n(A)$ which commutes with left multiplication: $a(\bar{x}T) = (a\bar{x})T$. In a familiar way, assign to any T a uniquely determined matrix (t_{ij}) over A $(1 \leq i, j \leq n)$ so that $\bar{x}T = (\sum_i x_i t_{i1}, \ldots, \sum_i x_i t_{in})$.

If p is a projection in $\mathbb{M}_n(A)$, then p is a mapping on $S_n(A)$ having its range as a sub-module of $S_n(A)$. Then two projections are orthogonal means their sub-module ranges are so. The C^*-algebra $\mathbb{M}_n(A)$ contains numerous projections. For each $a \in A$ and each pair of indices $i, j (i \neq j, \ 1 \leq i, j \leq n)$, H. Dye in [7] defined the projection $P_{i,j}(a)$ in $\mathbb{M}_n(A)$, whose range consists of all left multiples of the vector with 1 in the ith-place, a in the jth-place and zeros elsewhere. As a matrix

$$P_{i,j}(a) = \begin{pmatrix} 0 & \cdots & & \cdots & & \cdots & & \cdots & & \cdots & 0 \\ \hdotsfor{11} \\ 0 & \cdots & (1+aa^*)^{-1} & \cdots & (1+aa^*)^{-1}a & \cdots & 0 \\ \hdotsfor{11} \\ 0 & \cdots & a^*(1+aa^*)^{-1} & \cdots & a^*(1+aa^*)^{-1}a & \cdots & 0 \\ \hdotsfor{11} \\ 0 & \cdots & & \cdots & & \cdots & & \cdots & & \cdots & 0 \end{pmatrix}$$

Recall that (see [7], p.74) a system of matrix units of a unital C^*-algebra A is a subset $\{e_{i,j}^r\}, 1 \leq i, j \leq n$ and $1 \leq r \leq m$ of A, such that

$$e_{i,j}^r e_{j,k}^r = e_{i,k}^r, \ e_{i,j}^r e_{k,l}^s = 0 \text{ if } r \neq s \text{ or } j \neq k, \ (e_{i,j}^r)^* = e_{j,i}^r, \ \sum_{i,r}^{n,m} e_{i,i}^r = 1$$

and for every i, $e_{i,i} \in \mathcal{P}(A)$. For the C^*-complex matrix algebra $\mathbb{M}_n(\mathbb{C})$, let $\{E_{i,j}\}_{i,j=1}^n$ denote the standard system of matrix units of the algebra, that is $E_{i,j}$ is the $n \times n$ matrix over $\mathbb{C}$ with 1 at the place $i \times j$ and zeros elsewhere. It is also known that $\mathbb{M}_n(A)$ is $*$-isomorphic to $A \otimes \mathbb{M}_n(\mathbb{C})$ (see [11]). We will see that having a system of matrix units is a necessary condition in order that a C^*-algebra A is $*$-isomorphic to a C^*-matrix algebra $\mathbb{M}_n(B)$. Using the notion of a system of matrix units, we write

$$\begin{aligned} P_{i,j}(a) &= (1+aa^*)^{-1} \otimes E_{i,i} + (1+aa^*)^{-1}a \otimes E_{i,j} \\ &+ a^*(1+aa^*)^{-1} \otimes E_{j,i} + a^*(1+aa^*)^{-1}a \otimes E_{j,j} \in \mathcal{P}(\mathbb{M}_n(A)). \end{aligned}$$

If $a = 0$, then $P_{i,j}(a)$ is the ith diagonal matrix unit of $\mathbb{M}_n(A)$, which is $1 \otimes E_{i,i}$, or simply E_i.

Also in [10], M. Stone called the projection $P_{i,j}(a)$ the characteristics matrix of a.

H. Dye used these projections as a main tool to prove that an isomorphism between the discrete unitary groups of von Neumann factors not of type I_n, is implemented by a $*$-isomorphism between the factors themselves [[7], Theorem 2]. Indeed, let us recall main parts of his proof. Let A and B be two unital C^*-algebras and let $\varphi : \mathcal{U}(A) \to \mathcal{U}(B)$ be an isomorphism. As φ preserves self-adjoint unitaries, it induces a natural bijection $\theta_\varphi : \mathcal{P}(A) \to \mathcal{P}(B)$ between the sets of projections of A and B given by

$$1 - 2\theta_\varphi(p) = \varphi(1 - 2p), \ p \in \mathcal{P}(A).$$

This mapping is called a projection orthoisomorphism, if it preserves orthogonality, i.e. $pq = 0$ iff $\theta(p)\theta(q) = 0$.

Now, let θ be an orthoisomorphism from $\mathcal{P}(\mathbb{M}_n(A))$ onto $\mathcal{P}(\mathbb{M}_n(B))$. In [[7], Lemma 8] when A and B are von Neumann algebras, Dye proved that for any unitary $u \in \mathcal{U}(A)$, $\theta(P_{i,j}(u)) = P_{i,j}(v)$, for some unitary $v \in \mathcal{U}(B)$. A similar result is proved in the case of simple, unital C^*-algebras by the author in [1]. Afterwards, Dye in [[7], Lemma 6], proved that there exists a $*$-isomorphism (or $*$-antiisomorphism) from $\mathbb{M}_n(A)$ onto $\mathbb{M}_n(B)$ which coincides with θ on the projections $P_{i,j}(a)$. In fact, he proved that θ induces the $*$-isomorphism ϕ from A onto B defined by the relation $P_{i,j}(a) = P_{i,j}(\phi(a))$.

In this paper, we study the projections $P_{i,j}(a)$ of a C^*-matrix algebra $\mathbb{M}_n(A)$, for some C^*-algebra A, and we deduce main results concerning such projections.

The paper is organized as follows: In Section 2, we show that every projection in $\mathbb{M}_2(\mathbb{C})$ is of the form $P_{1,2}(a)$, for $a \in \mathbb{C}$. In Section 3, we show that some infinite C^*-algebra A is isomorphic to its matrix algebra $\mathbb{M}_n(A)$, such as the Cuntz algebra $\mathcal{O}_n$, so the projections $P_{i,j}(a)$ can be considered as projections of A.

In a simple, unital purely infinite C^*-algebra A, M. Leen proved that self-adjoint unitaries (also called $*$-symmetries, or involutions) generate the connected component $\mathcal{U}_0(A)$ of the unitary group $\mathcal{U}(A)$. Indeed, any unitary can be written as a product of eleven $*$-symmetries. In Section 4, we modify Leen's proof, and we write these $*$-symmetry factors explicitly. By revising his proof and fixing some arbitraries using a given system of matrix units, we show that eight of these $*$-symmetry factors are in fact of the form $1 - 2P_{i,j}(\omega)$, $\omega \in \mathcal{U}(A)$.

Finally, in Section 5, we compute the K_0-class of such certain projections, and we prove that in simple, unital purely infinite C^*-algebras (assuming $K_1 = 0$), all projections of the form $P_{i,j}(u)$, $u \in \mathcal{U}(A)$ have trivial K_0-class. As a good application for $\mathcal{O}_n$, we have that every unitary can be written as a product of eleven $*$-symmetries (self-adjoint unitaries, also called involutions), where eight of them are of the form $1 - 2P_{i,j}(\omega)$, $\omega \in \mathcal{U}(\mathcal{O}_n)$. Hence using [2] (Lemma 2.1), all such involutions of the form $1 - 2P_{i,j}(\omega)$ are indeed conjugate, as group elements in $\mathcal{U}(\mathcal{O}_n)$.

2   The 2×2-Complex Algebra Case

Let A be a unital C^*-algebra, and let $\mathcal{P}^n_{i,j}(A)$ denote the family of all projections in $\mathbb{M}_n(A)$ of the form $P_{i,j}(a)$, $1 \leq i, j \leq n$, $a \in A$. Also, let $\mathcal{U}^n_{i,j}(A)$ denote the set of all self-adjoint unitaries in $\mathbb{M}_n(A)$ of the form $1 - 2P_{i,j}(a)$, $1 \leq i, j \leq n$, $a \in A$. Notice that $\mathcal{P}^n_{i,j}(A)$ contains non-trivial projections. In this small section, we show that in the case of $\mathbb{M}_2(\mathbb{C})$, the set $\mathcal{P}^2_{i,j}(\mathbb{C})$ includes all the non-trivial projections $\mathcal{P}(\mathbb{M}_2(\mathbb{C}))$, i.e. every non-trivial projection is of the form $P_{i,j}(a)$, for some complex number a.

Proposition 2.1. *If* $p \in \mathcal{P}(\mathbb{M}_2(\mathbb{C})) \backslash \{0, 1\}$*, then* $p \in \mathcal{P}^2_{i,j}(\mathbb{C})$*.*

Proof. Let $p = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a non-trivial projection in $\mathcal{P}(\mathbb{M}_2(\mathbb{C}))$. Then a and d are real numbers. If $b = 0$, then p is either the diagonal matrix unit $E_{1,1}$ or $E_{2,2}$. Otherwise, we have $a + b = 1, a = a^2 + |b|^2$ and $d = d^2 + |b|^2$, therefore $|b|^2 \leq \frac{1}{4}$. By strightforward computations, one can deduce that p is of the form

$$P_{1,2}\left(\frac{2b}{1 + \sqrt{1 - 4|b|^2}} \right), \quad \text{or} \quad P_{1,2}\left(\frac{2b}{1 - \sqrt{1 - 4|b|^2}} \right).$$

$\square$

Remark 2.2. *The projections in* $\mathcal{P}^n_{i,j}(A)$ *are all of rank one by definition, this implies that in the case of* $\mathbb{M}_3(\mathbb{C})$*, the set* $\mathcal{P}^3_{i,j}(\mathbb{C})$ *does not cover all the non-trivial projections. Indeed, there are projections in* $\mathcal{P}(\mathbb{M}_3(\mathbb{C}))$ *of rank one which do not belong to* $\mathcal{P}^3_{i,j}(\mathbb{C})$*, since every projection in this latest family projects into a subspace of* $\mathbb{C}^3$ *which lies entirely in one coordinate plan.*

3   Some Results for infinite C^*-algebras

Let A be a unital C^*-algebra having a system of matrix units $\{e_{i,j}\}^n_{i,j=1}$, for some $n \geq 3$. Recall that $e_{1,1}Ae_{1,1}$ is a C^*-algebra (corner algebra) which has $e_{1,1}$ as a unit. This system of matrix units implements a $*$-isomorphism between A and $\mathbb{M}_n(e_{1,1}Ae_{1,1})$. Indeed, let us define the mapping

$$\eta_1 : \mathbb{M}_n(e_{1,1}Ae_{1,1}) \to A$$

by

$$\eta_1((a_{i,j})^n) = \sum_{i,j=1}^n e_{i,1}a_{i,j}e_{1,j}.$$

Moreover if $e_{1,1}$ is equivalent to 1 (i.e. A is assumed to be infinite C^*-algebra), then there exists a partial isometry v of A such that $v^*v = e_{1,1}$ and $vv^* = 1$, and this defines the $*$-isomorphism $\Delta_v : A \to e_{1,1}Ae_{1,1}$ by $\Delta_v(x) = v^*xv$. The isomorphism Δ_v can be used to decompose a projection as a sum of orthogonal equivalent projections.

Proposition 3.1. *Let A be a unital C^*-algebra having a system of matrix units $\{e_{i,j}\}_{i=1}^n$. If p is equivalent to the unity, then p can be written as a sum of orthogonal equivalent subprojections.*

Proof. As p equivalent to 1, we consider the isomorphism Δ_v, then apply it to the equality $1 = \sum_{i=1}^n e_{i,i}$, to get $p = \sum_{i=1}^n v^* e_{i,i} v$. Then $p_i = v^* e_{i,i} v$, for all $1 \leq i \leq n$, are equivalent subprojections of p. $\qquad\square$

Recall that, for two unital C^*-algebras A and B, if $\alpha : A \to B$ is a $*$-isomorphism, then α induces the $*$-isomorphism $\widehat{\alpha} : \mathbb{M}_n(A) \to \mathbb{M}_n(B)$, which is defined by $(a_{i,j}) \mapsto (\alpha(a_{i,j}))$. Then we have the following result.

Proposition 3.2. *Let A be an infinite unital C^*-algebra having a system of matrix units $\{e_{i,j}\}_{i,j=1}^n$. If $e_{1,1}$ is equivalent to 1, then $\mathbb{M}_n(A)$ is $*$-isomorphic to A.*

Proof. Let $\Delta_v : A \to e_{1,1} A e_{1,1}$ and $\eta_1 : \mathbb{M}_n(e_{1,1} A e_{1,1}) \to A$ be defined as above. Then the mapping $\eta = \eta_1 \circ \widehat{\Delta_v}$ is a $*$-isomorphism from $\mathbb{M}_n(A)$ onto A. Moreover,

$$\eta(a_{i,j})^n = \sum_{i,j}^n e_{i,1} v^* a_{i,j} v e_{1,j}, \text{ and}$$

$$\eta^{-1}(x) = (v e_{1,i} x e_{j,1} v^*)_{i,j}^n.$$

$\qquad\square$

As a main example of purely infinite C^*-algebras, let us recall the Cuntz algebra $\mathcal{O}_n$; $n \geq 2$, is the universal C^*-algebra which is generated by isometries $s_1, s_2, \ldots, s_n$, such that $\sum_{i=1}^n s_i s_i^* = 1$ with $s_i^* s_j = 0$, when $i \neq j$ and $s_i^* s_i = 1$ (for more details, see [5], [[6], p.149]). Let

$$e_{i,j} = s_i s_j^*, \qquad 1 \leq i, j \leq n . \tag{1}$$

Then $\{e_{i,j}\}_{i,j=1}^n$ forms a system of matrix units for $\mathcal{O}_n$. As s_1^* partial isometry between $e_{1,1}$ and the unity, then Proposition 3.2 shows that the mapping

$$\eta : \mathbb{M}_n(\mathcal{O}_n) \to \mathcal{O}_n, \quad (a_{i,j})_{i,j} \mapsto \sum_{i,j=1}^n s_i a_{i,j} s_j^* \tag{2}$$

is a $*$-isomorphism. Indeed, for $x \in \mathcal{O}_n$, $\eta^{-1}(x) = (s_i^* x s_j)_{i,j} \in \mathbb{M}_n(\mathcal{O}_n)$. Therefore, we have proved the following result, which is in fact known, but for sake of completeness:

Proposition 3.3. *The Cuntz algebra $\mathcal{O}_n$ is isomorphic to the C^*-algebra $\mathbb{M}_n(\mathcal{O}_n)$.*

Then for $a \in \mathcal{O}_n$, $P_{i,j}(a)$ are considered as projections of $\mathcal{O}_n$ by applying the mapping η. Therefore,

$$P_{i,j}(a) = s_i(1+aa^*)^{-1}s_i^* + s_i(1+aa^*)^{-1}as_j^* + s_j a^*(1+aa^*)^{-1}s_i^* + s_j a^*(1+aa^*)^{-1}as_j^*.$$

4 Unitary Factors in Purely Infinite C^*-Algebras

Recall that in a unital C^*-algebra A, every self-adjoint unitary u ($*$-symmetry, or also called an involution) can be written as $u = 1 - 2p$, for some projection $p \in \mathcal{P}(A)$, let us say " the self-adjoint unitary u is associated to the projection p". In this section, we assume that A is purely infinite simple C^*-algebra, and we study the factorizations of unitaries of A. Recall that in [9], M. Leen proved that every unitary in the connected component of the unity $\mathcal{U}_0(A)$ is generated by $*$-symmetries.

Consider a system of matrix units $\{e_{i,j}\}_{i,j=1}^n$ of A, with $e_{1,1} \sim 1$. Let us recall the $*$-isomorphisms $\eta_1 : \mathbb{M}_n(e_{1,1}Ae_{1,1}) \to A$, and $\eta = \eta_1 \circ \widehat{\Delta_v}$ from $\mathbb{M}_n(A)$ onto A. We modify Leens' proof of Theorem 3.5 in [9] by revising his arguments, and then we prove the following main theorem, which shows that every unitary of A can be factorized as a product of eleven self-adjoint unitaries ($*$-symmetries) moreover, where eight of such factors are associated to the projections $P_{i,j}(\mu)$, for some $\mu \in \mathcal{U}(A)$.

Theorem 4.1. *Let A be a simple, unital purely infinite C^*-algebra, such that $K_1(A) = 0$, and let $\{e_{i,j}\}_{i,j=1}^n$ be a system of matrix units of A, with $e_{1,1} \sim 1$. Then every unitary a of A can be written as*

$$a = z_1(\prod_{k=1}^4 v_k)z_2z_3,$$

where z_1, z_2, z_3 are some self-adjoint unitaries and the $v_i's$ are the self-adjoint unitaries of A defined by:

$$
\begin{aligned}
v_1 &= [1 - 2\eta(P_{1,2}(-\alpha))][1 - 2\eta(P_{1,2}(-1))] \\
v_2 &= [1 - 2\eta(P_{1,3}(-\alpha))][1 - 2\eta(P_{1,3}(-1))] \\
v_3 &= [1 - 2\eta(P_{1,2}(-\gamma))][1 - 2\eta(P_{1,2}(-1))] \\
v_4 &= [1 - 2\eta(P_{1,3}(-\gamma))][1 - 2\eta(P_{1,3}(-1))],
\end{aligned}
$$

for some $\alpha, \gamma \in \mathcal{U}(A)$.

Consequently, as the Cuntz algebra is simple, unital purely infinite C^*-algebra, and $K_1(\mathcal{O}_n) = 0$, see [4], and using Proposition 3.3, we have the following result.

Corollary 4.2. *If $u \in \mathcal{U}(\mathcal{O}_n)$, then*

$$
\begin{aligned}
u &= z_1(1 - 2P_{1,2}(-\alpha))(1 - 2P_{1,2}(-1))(1 - 2P_{1,3}(-\alpha))(1 - 2P_{1,3}(-1)) \\
&\quad \cdot (1 - 2P_{1,2}(-\gamma))(1 - 2P_{1,2}(-1))(1 - 2P_{1,3}(-\gamma))(1 - 2P_{1,3}(-1))z_2z_3,
\end{aligned}
$$

for some self-adjoint unitaries z_1, z_2, z_3 and $\alpha, \gamma \in \mathcal{U}(\mathcal{O}_n)$.

Now, in order to prove out main theorem, let us recall the following result of M. Leen.

Theorem 4.3 ([9], Theorem 3.8). *Let A be a simple, unital purely infinite C^*-algebra. Then the $*$-symmetries (self-adjoint unitaries) generate the connected component of the unity $\mathcal{U}_0(A)$.*

So Leen proved that every unitary in the component of the unity, can be written as a finite product of self-adjoint unitaries. We shall use Leen's approach, indeed, we fix some arbitrates, and we modify some of his arguments. Then using the system of matrix units and the mappings η_1, η, we write some arguments in an explicit way. Finally, we deduce that eight of those self-adjoint unitaries, as factors, are in fact associated to the projections $P_{i,j}(u)$, for some $u \in \mathcal{U}(A)$.

Let us introduce the following lemma which in fact, M. Leen used in his proof, and we do in our proof as well.

Lemma 4.4. *Let A be a simple, unital purely infinite C^*-algebra, and let ρ be a non-trivial projections of A. If $a \in \mathcal{U}_0(A)$, then there exist self-adjoint unitaries z_1, z_2, z_3 of A and $x \in \mathcal{U}_0(A)$ such that*

$$z_1 a z_2 z_3 = \begin{pmatrix} x & 0 \\ 0 & 1-\rho \end{pmatrix}.$$

Proof. Mimic the first part of the proof of Theorem 3.5 in [9], with replacing symmetries by $*$-symmetries and invertible by unitaries. $\qed$

Proof of Theorem 4.1:

Proof. Since A is a simple, unital purely infinite C^*-algebra, using [4], we have $K_1(A) \simeq \mathcal{U}(A)/\mathcal{U}_0(A)$. As $K_1(A)$ is assumed to be trivial, we have $\mathcal{U}(A) = \mathcal{U}_0(A)$. Now suppose $a \in \mathcal{U}(A)$, we shall revise Leen's proof, for many details, we just refer to him, and we explain new arguments which shall lead to our result. Let $p = e_{1,1}$, as $p \sim 1$, use Proposition 3.1 and the isomorphism Δ_u ($u^*u = e_{1,1}, uu^* = 1$) to find a projection $p_1 < p$ (precisely, $p_1 = u^*e_{1,1}u$) which is equivalent to p moreover, set the partial isometry $v = u^*e_{1,1}$, and put $\rho = p - p_1$. Using Lemma 4.4, there exist self-adjoint unitaries z_1, z_2 and z_3 such that

$$z_1 a z_2 z_3 = \begin{pmatrix} x & 0 \\ 0 & 1-\rho \end{pmatrix},$$

where $x \in \mathcal{U}(\rho A \rho)$. We will show that the right hand side can be written as a product of eight self-adjoint unitaries, each of them is associated to a projection of the form $\eta P_{i,j}(u)$, for some $u \in \mathcal{U}(A)$. We may replace $z_1 a z_2 z_3$ by a.

Choose $q = e_{2,2}$, $r = e_{3,3}$ and put $r_1 = p+q+r$, then we have $q \sim r < 1-p-q$. Let $v_1 = e_{2,1}$, $v_2 = e_{3,2}$, and $v_3 = e_{1,3}$, so v_1, v_2 and v_3 are partial isometries such that

$$v_1^*v_1 = p, \ v_1v_1^* = q, \ v_2^*v_2 = q, \ v_2v_2^* = r, \ v_3^*v_3 = r, \text{ and } v_3v_3^* = p.$$

Let $w = v_1 + v_2 + vv_3$. Recall that $\mathbb{K}$ denotes the compact operators on the separable, infinite dimensional Hilbert space $\ell^2(\mathbb{N})$. By I in $\rho A \rho \otimes \mathbb{K}$ we mean $\rho \otimes 1_\infty(\mathbb{C})$.

Leen defined in his proof three isomorphisms: $\rho A \rho \otimes \mathbb{K} \longrightarrow r_1 A r_1$. In order to build the first of the three copies of $\rho A \rho \otimes \mathbb{K}$, he defined an infinite collection of projections using w and ρ as follows: $\rho_k = w \rho_{k-1} w^*$, for $k \geq 2$, $\rho_1 = \rho$ and $w_k = w^{k-1} \rho$. Then $w_k w_k^* = \rho_k$ and $w_k^* w_k = \rho$, the $\rho_k's$ are orthogonal equivalent projections which satisfy $\rho_{3n-2} < p$, $\rho_{3n-1} < q$ and $\rho_{3n} < r$, for $n \geq 1$.

Define $\chi : \rho A \rho \otimes \mathbb{K} \to r_1 A r_1$ by $y \otimes E_{i,j}(\mathbb{C}) \mapsto w_i y w_j^*$, and $I \mapsto r_1$. Next we produce two other copies of $\rho A \rho \otimes \mathbb{K}$ in $r_1 A r_1$ as follows: For each n choose orthogonal equivalent projections $\{e_{3n-2}^j : j = 1, \ldots, 4^{n-1}\}$ such that $e_{3n-2}^j \sim \rho_{3n-2}$ and

$$\rho_{3n-2} = \sum_{j=1}^{4^n-1} e_{3n-2}^j,$$

then put $e_{3n-1}^j = w(e_{3n-2}^j)w^*$ and $e_{3n}^j = w(e_{3n-1}^j)w^*$, for each n and j, and order the e_i^j's as: $e_1^1, e_2^1, e_3^1, e_4^1, \ldots e_4^4, e_5^1, \ldots$. Use the partial isometries which implements the equivalences $\rho_{3n-2} \sim e_{3n-2}^j$ and $\rho_{3n-2} \sim \rho$ to define partial isometries r_{3n-2}^j so that $r_{3n-2}^j (r_{3n-2}^j)^* = \rho$ and $(r_{3n-2}^j)^* r_{3n-2}^j = e_{3n-2}^j$, and put $r_{3n-1}^j = r_{3n-2}^j w^*$ and $r_{3n}^j = r_{3n-1}^j w^*$. Then use the r_i^j to define $\varphi_1 : \rho A \rho \otimes \mathbb{K} \to r_1 A r_1$.

Similarly choose orthogonal equivalent projections $\{f_i^j\}$ such that $\rho = f_1^1$ and

$$\rho_{3n-1} = \sum_{j=1}^{2.4^{n-1}} f_{3n-1}^j,$$

for $n \geq 1$. Then put $f_{3n}^j = w(f_{3n-1}^j)w^*$ and $f_{3n+1}^j = w(f_{3n}^j)w^*$, for any n and j. Order the f_i^j as:

$$f_1^1, f_2^1, f_2^2, f_3^1 f_3^2, f_4^1, f_4^2, f_5^1, \ldots, f_5^8, f_6^1, \ldots.$$

Using the partial isometries which implement $f_i^j \sim \rho$, define $\varphi_2 : \rho A \rho \otimes \mathbb{K} \to r_1 A r_1$.

Recall that $w = e_{2,1} + e_{3,2} + u^* e_{1,3}$, then

$$w^2 = e_{2,1} u^* e_{1,3} + e_{3,1} + e_{3,2} u^* e_{1,3} + u^* e_{1,2} + u^* e_{1,3} u^* e_{1,3}$$

Now for $1 \leq k \leq 3$, let $u_k = w^{k-1} p$ therefore $u_k = e_{k,1}$. Define the map

$$\zeta : r_1 A r_1 \longrightarrow \mathbb{M}_3(pAp)$$

$$\text{by } x \longmapsto (u_i^* x u_j)_{i,j=1}^3$$

$$\text{i.e. } x \longmapsto (e_{1,i} x e_{j,1})_{i,j=1}^3.$$

The map ζ is a $*$-isomorphism, indeed

$$\zeta^{-1} : \mathbb{M}_3(pAp) \longrightarrow r_1 A r_1$$

$$\text{is defined by } (a_{i,j}) \longmapsto \sum_{i,j}^3 e_{i,1} a_{i,j} e_{1,j}.$$

Now we turn to factorization of a (In fact, we factorize $z_1 a z_2 z_3$). Let $r_0 = 1 - r_1$. From the definitions of φ_i's, and since $a = \begin{pmatrix} x & 0 \\ 0 & 1-\rho \end{pmatrix}$, where $x \in \mathcal{U}(\rho A \rho)$, we have the following:

$$
\begin{aligned}
\varphi_i(\operatorname{diag}(x,1,1,\ldots)) &= \varphi_i(\operatorname{diag}(x-\rho,0,0,\ldots)+I) \\
&= r_1 + \varphi_i(\operatorname{diag}(x-\rho,0,0,\ldots)) \\
&= r_1 + x - \rho \\
&= p + q + r + x - \rho \\
&= a - r_0.
\end{aligned}
$$

If $a - r_0$ is a product of $*$-symmetries in $r_1 A r_1$, then a is a product of $*$-symmetries in A. Using [[9], proof of Theorem 1], we factorize $\operatorname{diag}(x,1,1,\ldots)$ as follows:

$\operatorname{diag}(x,1,1,\ldots) =$
$\operatorname{diag}(x^{1/2}, x^{-1/2}, 1, x^{1/8}, x^{1/8}, x^{1/8}, x^{1/8}, x^{-1/8}, x^{-1/8}, x^{-1/8}, x^{-1/8}, 1, 1, 1, 1, \ldots)$
$.\operatorname{diag}(x^{1/2}, 1, x^{-1/2}, x^{1/8}, x^{1/8}, x^{1/8}, x^{1/8}, 1, 1, 1, 1, x^{-1/8}, x^{-1/8}, x^{-1/8}, x^{-1/8}, \ldots)$
$.\operatorname{diag}(1, x^{1/4}, x^{1/4}, x^{-1/4}, x^{-1/4}, 1, 1, x^{1/16}, x^{1/16}, x^{1/16}, x^{1/16}, x^{1/16}, x^{1/16}, x^{1/16}, x^{1/16}, x^{-1/16}, \ldots)$
$.\operatorname{diag}(1, x^{1/4}, x^{1/4}, 1, 1, x^{-1/4}, x^{-1/4}, x^{1/16}, x^{1/16}, x^{1/16}, x^{1/16}, x^{1/16}, x^{1/16}, x^{1/16}, x^{1/16}, \ldots)$
$= b_1 b_2 b_3 b_4.$

We must factorize b_i as a product of $*$-symmetries. We use φ_1 to factorize b_1, b_2, and use φ_2 to factorize b_3, b_4. We check the details only for b_1 and b_2. Let us first factorize b_1.

$$
b_1 = (b_1^1, b_1^2, \ldots, b_1^n, \ldots);
$$

where $b_1^n = \operatorname{diag}(x_n, x_n^{-1}, 1)$ and x_n be the diagonal $4^{n-1} \times 4^{n-1}$ matrix with all diagonal entries equal to $x^{\left(\frac{1}{2 \cdot 4^{n-1}}\right)}$, so $b_1 \in \prod_{n=1}^{\infty} \mathbb{M}_3(\mathbb{M}_{4^{n-1}}(\rho A \rho))$. Then Leen defined the map

$$
\Phi : \prod_{n=1}^{\infty} \mathbb{M}_3(\mathbb{M}_{4^{n-1}}(\rho A \rho)) \longrightarrow \prod_{n=1}^{\infty} \mathbb{M}_3(\rho A \rho)
$$

Let $\Phi(b_1) = c^1$. He showed that $\chi(c^1) = \varphi_1(b_1)$, and

$$
\zeta(\chi(c^1)) = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha^{-1} & 0 \\ 0 & 0 & p \end{pmatrix};
$$

where α is a unitary in pAp. Let $\beta_1 = \begin{pmatrix} 0 & \alpha & 0 \\ \alpha^{-1} & 0 & 0 \\ 0 & 0 & p \end{pmatrix}$ and $\beta_2 = \begin{pmatrix} 0 & p & 0 \\ p & 0 & 0 \\ 0 & 0 & p \end{pmatrix}$,

so $\beta_1 \beta_2 = \zeta(\chi(c^1))$ and

$$
\frac{I - \beta_1}{2} = P_{1,2}(-\alpha), \quad \frac{I - \beta_2}{2} = P_{1,2}(-p),
$$

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where now $P_{1,2}(-\alpha), P_{1,2}(-p) \in \mathcal{P}(\mathbb{M}_3(pAp))$. Therefore,

$$\chi(c^1) = \zeta^{-1}(\beta_1)\zeta^{-1}(\beta_2) = (r_1 - 2\zeta^{-1}(P_{1,2}(-\alpha)))(r_1 - 2\zeta^{-1}(P_{1,2}(-p))),$$

but $\zeta^{-1}(P_{1,2}(-\alpha)) = \eta_1(P_{1,2}(-\alpha))$ and $\zeta^{-1}(P_{1,2}(-p) = \eta_1(P_{1,2}(-p))$.
Now to factorize b_2:

$$b_2 = (b_2^1, b_2^2, \ldots, b_2^n, \ldots) \text{ where } b_2^n = \operatorname{diag}(x_n, 1, x_n^{-1})$$

and x_n is the same as in b_1 so $b_2 \in \prod_{n=1}^\infty \mathbb{M}_3(\mathbb{M}_{4^{n-1}}(\rho A\rho))$. Let $\Phi(b_2) = c^2$.
$\chi(c^2) = \varphi_1(b_2)$

$$\zeta(\chi(c^2)) = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & \alpha^{-1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \alpha \\ 0 & p & 0 \\ \alpha^{-1} & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & p \\ 0 & p & 0 \\ p & 0 & 0 \end{pmatrix} = \beta_3\beta_4$$

so β_3, β_4 are self-adjoint unitaries in $\mathbb{M}_3(pAp)$, indeed

$$\frac{I - \beta_3}{2} = P_{1,3}(-\alpha), \text{ and } \frac{I - \beta_4}{2} = P_{1,3}(-p)$$

therefore,

$$\chi(c^2) = \zeta^{-1}(\beta_3)\zeta^{-1}(\beta_4) = (r_1 - 2\zeta^{-1}(P_{1,3}(-\alpha)))(r_1 - 2\zeta^{-1}(P_{1,3}(-p)))$$

but $\zeta^{-1}(P_{1,3}(-\alpha)) = \eta_1(P_{1,3}(-\alpha))$, and $\zeta^{-1}(P_{1,3}(-p)) = \eta_1(P_{1,3}(-p))$.

Now we use φ_2 to factorize b_3 and b_4:

$$b_3 = (1, b_3^1, b_3^2, \ldots, b_3^n, \ldots); \text{ where } b_3^n = \operatorname{diag}(x_n, x_n^{-1}, 1)$$

and x_n is a $2.4^{n-1} \times 2.4^{n-1}$ diagonal of diagonal entries matrix $x^{\frac{1}{4.4^{n-1}}}$
so $b_3 \in (\rho A\rho) \times (\prod \mathbb{M}_3(\mathbb{M}_{2.4^{n-1}}(\rho A\rho)))$. Then we define the map

$$\Phi' : (\rho A\rho) \times (\prod \mathbb{M}_3(\mathbb{M}_{2.4^{n-1}}(\rho A\rho))) \longrightarrow (\rho A\rho) \otimes \mathbb{K},$$

which acts as the identity map on the first component. Let $\Phi'(b_3) = d^1$. We
have $\chi(d^1) = \varphi_2(b_3)$.

$$\zeta(\chi(d^1)) = \begin{pmatrix} \gamma & 0 & 0 \\ 0 & \gamma^{-1} & 0 \\ 0 & 0 & p \end{pmatrix};$$

where γ is a unitary in $\rho A\rho$, so similar to case b_1, just replace α by γ, to get

$$\chi(d^1) = (r_1 - 2\eta_1(P_{1,2}(-\gamma)))(r_1 - 2\eta_1(P_{1,2}(-p))).$$

Now finally to factorize b_4:

$$b_4 = \operatorname{diag}(1, b_4^1, b_4^2, \ldots, b_4^n, \ldots); \text{ where } b_4^n = \operatorname{diag}(x_n, 1, x_n^{-1}),$$

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and x_n is the same as in the case of b_3 Let $\Phi'(b_4) = d^2$. We have $\chi(d^2) = \varphi_2(b_4)$.

$$\zeta(\chi(d^2)) = \begin{pmatrix} \gamma & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & \gamma^{-1} \end{pmatrix}$$

again, it's similar to case b_2, so

$$\chi(d^2) = (r_1 - 2\eta_1(P_{1,3}(-\gamma)))(r_1 - 2\eta_1(P_{1,3}(-p))).$$

Then, we factorize $a - r_0$ as

$$a - r_0 = \chi(c^1)\chi(c^2)\chi(d^1)\chi(d^2)$$

therefore,

$$\begin{pmatrix} a - r_0 & 0 \\ 0 & r_0 \end{pmatrix} = \begin{pmatrix} \chi(c^1) & 0 \\ 0 & r_0 \end{pmatrix}\begin{pmatrix} \chi(c^2) & 0 \\ 0 & r_0 \end{pmatrix}\begin{pmatrix} \chi(d^1) & 0 \\ 0 & r_0 \end{pmatrix}\begin{pmatrix} \chi(d^2) & 0 \\ 0 & r_0 \end{pmatrix}.$$

And also we have the following:

$$\begin{pmatrix} \chi(c^1) & 0 \\ 0 & r_0 \end{pmatrix} = \begin{pmatrix} r_1 - 2\eta_1(P_{1,2}(-\alpha)) & 0 \\ 0 & r_0 \end{pmatrix}\begin{pmatrix} r_1 - 2\eta_1(P_{1,2}(-p)) & 0 \\ 0 & r_0 \end{pmatrix}$$

$$\begin{pmatrix} \chi(c^2) & 0 \\ 0 & r_0 \end{pmatrix} = \begin{pmatrix} r_1 - 2\eta_1(P_{1,3}(-\alpha)) & 0 \\ 0 & r_0 \end{pmatrix}\begin{pmatrix} r_1 - 2\eta_1(P_{1,3}(-p)) & 0 \\ 0 & r_0 \end{pmatrix}$$

$$\begin{pmatrix} \chi(d^1) & 0 \\ 0 & r_0 \end{pmatrix} = \begin{pmatrix} r_1 - 2\eta_1(P_{1,2}(-\gamma)) & 0 \\ 0 & r_0 \end{pmatrix}\begin{pmatrix} r_1 - 2\eta_1(P_{1,2}(-p)) & 0 \\ 0 & r_0 \end{pmatrix}$$

$$\begin{pmatrix} \chi(d^2) & 0 \\ 0 & r_0 \end{pmatrix} = \begin{pmatrix} r_1 - 2\eta_1(P_{1,3}(-\gamma)) & 0 \\ 0 & r_0 \end{pmatrix}\begin{pmatrix} r_1 - 2\eta_1(P_{1,3}(-p)) & 0 \\ 0 & r_0 \end{pmatrix}.$$

Therefore,

$$\begin{aligned} z_1 a z_2 z_3 &= (1 - 2\eta_1(P_{1,2}(-\alpha)))(1 - 2\eta_1(P_{1,2}(-p)))(1 - 2\eta_1(P_{1,3}(-\alpha)))(1 - 2\eta_1(P_{1,3}(-p))) \\ &\quad \cdot \; (1 - 2\eta_1(P_{1,2}(-\gamma)))(1 - 2\eta_1(P_{1,2}(-p)))(1 - 2\eta_1(P_{1,3}(-\gamma)))(1 - 2\eta_1(P_{1,3}(-p))) \end{aligned}$$

The factors in the right side are all self-adjoint unitaries in A. Hence using the mapping η, we have that

$$\begin{aligned} a &= z_1(1 - 2\eta(P_{1,2}(-\alpha)))(1 - 2\eta(P_{1,2}(-1)))(1 - 2\eta(P_{1,3}(-\alpha)))(1 - 2\eta(P_{1,3}(-1))) \\ &\quad \cdot \; (1 - 2\eta(P_{1,2}(-\gamma)))(1 - 2\eta(P_{1,2}(-1)))(1 - 2\eta(P_{1,3}(-\gamma)))(1 - 2\eta(P_{1,3}(-1)))z_2 z_3 \end{aligned}$$

where α and γ are unitaries in A, and this ends the proof. $\qquad\square$

Finally, let us finish this section by the following open question:

Question 4.5. *In the Cuntz algebra $\mathcal{O}_n$, do self-adjoint unitaries of the form $\{1 - 2P_{i,j}(a)\}$ generate the unitary group $\mathcal{U}(\mathcal{O}_n)$?*

5 K-Theory of Certain Projections

In this section, we study the K_0-class of the projections $P_{i,j}(u)$, where u is a unitary of some unital C^*-algebra A. In particular, if A is a simple purely infinite C^*-algebra, with $K_1(A) = 0$, or A is a von Neumann factor of type II_1, or III, then for any unitary u of A, $P_{i,j}(u)$ has trivial K_0-class. Afterwards, we present an application of Theorem 4.1, to the case of Cuntz algebras.

Proposition 5.1. *Let A be a unital C^*-algebra. If v is a unitary in A of finite order, then $[P_{i,j}(v)] = [1]$ in $K_0(A)$.*

Proof. Consider a unitary v in A, such that $v^m = 1$, for some positive integer m. For $i \neq j$, let

$$W = \frac{1}{\sqrt{2}}(v \otimes E_{i,i} + v \otimes E_{i,j} + E_{j,i} - E_{j,j} + \sum_{k \notin \{i,j\}} \sqrt{2} \otimes E_{k,k}) \ ,$$

then $W^* = \frac{1}{\sqrt{2}}(v^{m-1} \otimes E_{i,i} + E_{i,j} + v^{m-1} \otimes E_{j,i} - E_{j,j} + \sum_{k \notin \{i,j\}} \sqrt{2} \otimes E_{k,k})$, therefore $W \in \mathcal{U}(\mathbb{M}_n(A))$. Moreover,

$$
\begin{aligned}
W^* P_{i,j}(v) W &= \frac{1}{4}(2v^{m-1} \otimes E_{i,i} + 2 \otimes E_{i,j})(\sqrt{2}W) \\
&= \begin{pmatrix} 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} \quad \text{(1 at the i-th place)} \\
&= E_{i,i}.
\end{aligned}
$$

This implies that the projection $P_{i,j}(v)$ is unitarily equivalent to $E_{i,i}$ in $\mathbb{M}_n(A)$, therefore we have that $[P_{i,j}(v)] = [1]$ in $K_0(A)$, hence the proposition has been checked. $\square$

Proposition 5.2. *Let A be a unital C^*-algebra. If w_1, w_2 and v are unitaries of A such that v has order m, then $[P_{i,j}(w_1 v w_2)] = [1]$ in $K_0(A)$.*

Proof. As w_1 and w_2 are unitaries in A, then for all $i \neq j$, $W = w_1 \otimes E_{i,i} + w_2^* \otimes E_{j,j} + \sum_{k \notin \{i,j\}} E_{k,k} \in \mathcal{U}(\mathbb{M}_n(A))$. Moreover, $W P_{i,j}(v) W^* = P_{i,j}(w_1 v w_2)$, therefore by Proposition (5.1) we have $[P_{i,j}(w_1 v w_2)] = [P_{i,j}(v)] = [1]$. $\square$

Proposition 5.3. *Let A be a unital C^*-algebra. If u and v are self-adjoint unitaries in A, then $[P_{i,j}(uv)] = [1]$ in $K_0(A)$.*

Proof. For $i \neq j$, let

$$W = \frac{1}{\sqrt{2}}(uv \otimes E_{i,i} + uv \otimes E_{i,j} + E_{j,i} - E_{j,j} + \sum_{k \notin \{i,j\}} \sqrt{2} \otimes E_{k,k}) \ ,$$

then $W \in \mathcal{U}(\mathbb{M}_n(A))$. Moreover,

$$
\begin{aligned}
W^* P_{i,j}(uv) W &= \frac{1}{4}(2uv \otimes E_{i,i} + 2 \otimes E_{i,j})(\sqrt{2}W) \\
&= E_{i,i},
\end{aligned}
$$

and this implies that the projection $P_{i,j}(uv)$ is unitarily equivalent to $E_{i,i}$ in $\mathbb{M}_n(A)$, therefore we have that $[P_{i,j}(uv)] = [1]$ in $K_0(A)$, hence the proposition has been checked. $\qquad\square$

Combining the previous results, we have the following theorem concerning the K_0-class of those projections $P_{i,j}(u)$ in $\mathcal{P}(\mathbb{M}_n(A))$, evaluated at any unitary u of A.

Theorem 5.4. *Let A be a simple, unital purely infinite C^*-algebra, such that $K_1(A)$ is the trivial group. If $u \in \mathcal{U}(A)$, then $[P_{i,j}(u)] = [1]$ in $K_0(A)$.*

Proof. Consider a unitary u of A. As $K_1(A) = 0$, and we know by [[4], p.188] that $K_1(A) \simeq \mathcal{U}(A)/\mathcal{U}_0(A)$ then using M. Leen's result (Theorem 4.3), we have that $u = \prod_{k=1}^{n} v_k$, where v_k is a self-adjoint unitary ($*$-symmetry) of A. If $n = 1$, then the result holds by using Proposition (5.1). Proposition (5.3) proves the case $n = 2$. If $n \geq 3$, then the result is done by Proposition (5.2), hence the proof is completed. $\qquad\square$

Moreover, as M. Broise in [[3], Theorem 1] proved that in the case of von Neumann factors of either type II_1 or III, the unitaries are generated by the self-adjoint unitaries, then a similar result in the case of von Neumann factors can be deduced as follows:

Theorem 5.5. *Let A be a von Neumann factor of type II_1 or III. If $u \in \mathcal{U}(A)$, then $[P_{i,j}(u)] = [1]$ in $K_0(A)$.*

Proof. Let u be a unitary of A. By [[3], Theorem 1], u can be written as a finite product of self-adjoint unitaries of A, then mimic the proof of Theorem 5.4. $\qquad\square$

Consequently, we have the following results concerning the K_0-class of some certain projections.

Corollary 5.6. *Let A be a unital C^*-algebra which is either:*
(1) Simple, purely infinite, with $K_1(A) = 0$, or
(2) von Neumann factor of type II_1, or III.
If u be a unitary of A, and p is the projection of $\mathbb{M}_n(A)$ defined by

$$
p = \frac{1}{2} \otimes E_{1,1} + \frac{v}{2} \otimes E_{1,2} + \frac{v^*}{2} \otimes E_{2,1} + \frac{1}{2} \otimes E_{2,2} + E_{3,3} + E_{4,4} \cdots + E_{m,m}
$$

for some positive integer $m \leq n - 2$, then $[p] = (m + 1)[1]$, in $K_0(A)$.

Proof. As the projection p is the orthogonal sums of $P_{1,2}(v) + E_{3,3} + E_{4,4} \cdots + E_{m,m}$, then by either Theorem 5.4 or 5.5,

$$[p] = [1] + ([1] + \cdots + [1]) = (m+1)[1].$$

$\square$

Corollary 5.7. *Let A be a unital C^*-algebra which is either:*
(1) Simple, purely infinite, with $K_1(A) = 0$, or
(2) von Neumann factor of type II_1, or III.
If $v_1, v_2 \cdots v_n$ are unitaries of A, and p is the projection of $\mathbb{M}_{2n}(A)$ defined by

$$
\begin{aligned}
p \quad = \quad & \frac{1}{2} \otimes E_{1,1} + \frac{v_1}{2} \otimes E_{1,2} + \frac{v_1^*}{2} \otimes E_{2,1} + \frac{1}{2} \otimes E_{2,2} \\
+ \quad & \frac{1}{2} \otimes E_{3,3} + \frac{v_2}{2} \otimes E_{3,4} + \frac{v_2^*}{2} \otimes E_{4,3} + \frac{1}{2} \otimes E_{4,4} + \cdots \\
+ \quad & \frac{1}{2} \otimes E_{2n-1,2n-1} + \frac{v_n}{2} \otimes E_{2n-1,2n} + \frac{v_n^*}{2} \otimes E_{2n,2n-1} + \frac{1}{2} \otimes E_{2n,2n},
\end{aligned}
$$

then $[p] = n[1]$, in $K_0(A)$.

Proof. Using Theorem 5.4 (or Theorem 5.5), we have

$$[p] = [P_{1,2}(v_1)] + [P_{3,4}(v_2) + \cdots + [P_{2n-1,2n}(v_n)] = n[1].$$

$\square$

Now let us prove the following lemma, which will be used in order to prove our main result in this section (Theorem 5.9), which is in fact a consequence application of Theorem 4.1, to the case of Cuntz algebras $\mathcal{O}_n$.

Lemma 5.8. *Let A be a unital, simple purely infinite C^*-algebra, with $K_1(A) = 0$, and let $\{e_{i,j}\}^n$, with $e_{1,1} \sim 1$ be a system of matrix units of A. Then for any unitary $u \in \mathcal{U}(A)$ we have $[\eta(P_{i,j}(u))] = [1]$ in $K_0(A)$.*

Proof. As we have seen in the proof of Propositions 5.1, 5.2, 5.3 and Theorem 5.4, there exists a unitary $W \in \mathcal{U}(\mathbb{M}_n(A))$, such that $W^* P_{i,j}(u) W = E_{i,i}$. Therefore,

$$\eta(W)^* \eta(P_{i,j}(u)) \eta(W) = \eta(E_{i,i}) = \eta_1 \hat{\Delta}_v(E_{i,i}) = \eta_1(e_{1,1} \otimes E_{i,i}) = e_{i,i}.$$

Then

$$\eta(P_{i,j}(u)) \sim_u e_{i,i} \sim e_{1,1} \sim 1,$$

hence $\eta(P_{i,j}(u))$ and 1 have the same class in $K_0(A)$. $\square$

Finally, let us consider the case of the Cuntz algebra $\mathcal{O}_n$. Let u be a self-adjoint unitary (involution), so $u = 1 - 2p$, for some $p \in \mathcal{P}(\mathcal{O}_n)$. We recall the concept *type of involution* which is introduced by the author in [2], as follows: Since $K_0(\mathcal{O}_n) \simeq \mathbb{Z}_n$ (see [4]), then the type of u is defined to be the element

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$[p]$ in $K_0(\mathcal{O}_n)$. By ([2], Lemma 2.1), two involutions are conjugate as group elements in $\mathcal{U}(\mathcal{O}_n)$ iff they have the same type.

As a consequence of Theorem 4.1, and the results concerning the K_0-group of the projections $P_{i,j}(u)$, which are deduced in this section, we have the following result.

Theorem 5.9. *If u is a unitary of $\mathcal{O}_n$, then there exist self-adjoint unitaries z_1, z_2, z_3 and v_k, for $1 \le k \le 8$ such that*

$$u = z_1(\prod_{k=1}^{8} v_k)z_2 z_3, \tag{3}$$

$v_k \in \{1 - 2\eta P_{i,j}(\omega)\}$, $\omega \in \mathcal{U}(\mathcal{O}_n)$ consequently, all the v_k factors are conjugate involutions.

Proof. Using [4] and [5], the Cuntz algebra $\mathcal{O}_n$ is simple, unital purely infinite C^*-algebra with trivial K_1-group. Then the decomposition of u as in Equation 3 holds by Theorem 4.1, so the type of each involution v_k is $[\eta(P_{i,j}(w))]$, for some $1 \le i \ne j \le n$ and a unitary w, hence by Lemma 5.8, the type of v_k is 1. Then by [[2], Lemma 2.1], all these involutions are conjugate indeed, to the trivial involution -1. $\qquad\square$

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